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# Determinism and randomness in quantum dynamics

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**Abstract.** A class of models is considered where a finite or infinite quantum dynamical system in a stationary state is probed by sequences of observations acting on a specified finite subsystem. The whole set of such experiments is described by a time-ordered (causal) and stationary quantum correlation kernel. It is shown that any such kernel can be decomposed in a unique way into a convex combination of two kernels, here called the regular and singular components. A singular system has a strong deterministic property, the predictability of the future from the knowledege of the past is limited only by the inevitable indeterminism of quantum measurements. Furthermore, in this case the full set of correlation functions of arbitrary time order, and hence the dynamical system itself, is determined by the causal kernel. Finite systems and infinite systems satisfying the KMS condition at finite temperature are of this type. In the regular case the dynamics contains a shift, there is a genuine asymptotic randomness and the dynamical system cannot be reconstructed in a unique way from the causal kernel. Non-trivial quantum Markov processes are shown to belong to this class.

## 1. Introduction

A fundamental problem in the theory of dynamical systems is whether a particular system is uniquely defined by the observations we can make on it when these are restricted in some way. A prototype solution is the Kolmogorov construction where a stationary stochastic process is built from a set of compatible cylinder measures, each representing the observation of the outcomes in a finite number of instants [1, 2]. In this way one obtains a dynamical system consisting of a group of automorphisms of a probability space and the construction is essentially unique. When the system is ergodic then it can be reconstructed from almost any sample path. A similar result in quantum theory is the Wightman reconstruction theorem of relativistic quantum field theory [3].

An analogue of these two problems concerns quantum dynamical systems where only incomplete measurements are made. The simplest form of this situation is that where the system is decomposed into an observed system $\mathscr{S}$ and a reservoir $\mathscr{R}$. $\mathscr{S}$ interacts with $\mathscr{R}$ and the measuring instruments, while $\mathscr{R}$ is not directly observed (section 2). In this paper the nature of $\mathscr{R}$ is rather irrelevant while the system $\mathscr{S}$ is defined by a subalgebra $A_{\mathscr{S}}$ of the operator algebra $A$ of the total system $\mathscr{S}+\mathscr{R}$. From a reference state on $A$ are defined a set of correlation kernels (section 3), the elements of which are expectation values of products of time translates of operators in $A_{\mathscr{S}}$ with a finite number of time arguments. In this work only stationary states and kernels will be considered, and the singular *quantum correlation kernel* (QCK) is used to denote a compatible set of kernels of all orders. For a general scheme of this kind Accardi *et al*

[4] proved a non-cummutative version of the Kolmogorov theorem which gives a reconstructed dynamical system which is unique up to unitary equivalence. There is one feature of this construction which is less satisfactory than the classical version. In quantum theory the non-commutative nature of quantum measurements means that the time order of the operators are crucial. Physically accessible are only the elements of a QCK where the causal time order is respected (section 3), rather than the full set of arbitrary time order used in [4].

The construction given in section 6 shows that given a stationary causal QCK, having a set of properties specified in section 5, there is always a minimal reconstruction which gives a W*-algebra in a Hilbert space, a strongly continuous group of unitary transformations representing the dynamics and a stationary state. However, in general this reconstruction is not unique up to unitary equivalence, which means that the correlations of arbitrary time order are not uniquely determined. It is shown in theorem 1 of section 6 that the QCK can be decomposed in a unique way into a convex combination of two QCKs, called here the regular and the singular part. A singular QCK gives a reconstructed system which is unique up to unitary equivalence. A regular QCK is characterized by a dynamics which contains a bilateral shift, and here the reconstruction always involves an arbitrary choice. The words regular and singular as used here are borrowed from the theory of stationary stochastic processes, there is no reference to the regularity of a dynamical system used as a contrast to chaos.

In the commutative case a stationary stochastic process can be decomposed in a unique way into a sum of two independent stationary processes, a deterministic or singular process and a purely indeterministic or regular process [5, 6]. For a singular process the whole future can be predicted with certainty from a complete knowledge of the past using a linear predictor. For a regular process such a prediction of the infinitely distant future is restricted to a trivial knowledge of the expectation value of the random variable. The results of section 6 form a non-commutative counterpart to this theory. However, there are significant differences between the commutative and non-commutative theories, and some of these are briefly discussed in section 8. In the quantum case the concept of predictability must be defined with some care as the outcomes of observations on any quantum system have a non-deterministic property which is independent of the dynamics. In theorem 2 in section 6 a concept of quantum determinism is formulated which gives to a system with a singular QCK as much predictability as quantum theory allows. This is in contrast with systems with a regular QCK, where the time evolution will irreversibly destroy some of the information contained in the past history, though not necessarily all of it. Here there is a true indeterminism coming from the dynamics.

One may wonder if there are many physically realistic open quantum systems with the strong predictability property discussed in theorem 2. The answer is that it is more difficult to find quantum systems which are truly unpredictable. The QCK is singular if the dynamics of the system (more precisely of the covariant representation defined in section 2) contains no shift. This is a property which is shared by a large class of quantum models (section 4). First, there are typical finite systems with an energy spectrum bounded below. Second, it holds for infinite systems when the stationary reference state defining the QCK satisfies the KMS condition which is a characteristic of thermal equilibrium at a finite temperature. In order to obtain a regular QCK we need a reservoir capable of supplying a quantum counterpart of white noise, and this corresponds to having an infinite temperature (section 7).

A special (zero temperature) case of singular systems is that where the stationary state is a ground state. This is the situation in relativistic QFT where the vacuum is the invariant state used to define the Wightman functions (vacuum expectation values). These functions are certainly determined by a subset of values which have a partial causal order analogous to that of the QCKs discussed here but defined by the Minkowski metric. However, it is known that they actually have the much stronger property of being uniquely determined by their values on so called Jost points ([3] section 2.4). These are $n$-tuples of space-time vectors where all difference arguments are spacelike vectors, such that the corresponding field operators commute or anticommute.

A reconstruction theorem using the causal QCK only was proved by Belavkin [7], who obtained unicity up to unitary equivalence without a restriction to singular QCKs. This achieved by allowing an isometric rather than a unitary representation of the dynamics and a degenerate representation of $A_{\mathscr{S}}$ which maps the unit operator on a projection. There is no contradiction with the results of this paper, here we need a unitary dilation of the isometric dynamics. The construction in section 6 differs from that of [7] in other details, and it introduces less structure in order to keep the formalism as simple as possible.

A special type of QCKs are those belonging to quantum Markov processes. These are generated by semigroups of completely positive maps with a normal stationary state in the way described in (7.1). This kind of process has a true asymptotic randomness when these maps are not unitary, and in this non-trivial case the QCK will be regular (theorem 3 of section 7). It turns out that the QCK is ergodic if and only if the stationary state is extremal.

## 2. Quantum dynamical systems

Consider a finite quantum system $\mathscr{S}$ described by a W*-algebra $A_{\mathscr{S}}=B(H_{\mathscr{S}})$, where $H_{\mathscr{S}}$ is a separable Hilbert space. This system is *open*, i.e. it is part of a larger, perhaps infinite, system. This is represented by $A_{\mathscr{S}}$ being a subalgebra of the algebra $A$ representing the whole system. When the whole system is finite $A$ can be chosen to be a W*-algebra. Recall that the GNS construction associates with any state $\rho$ a $*$-representation $\pi(A)$ in a Hilbert space $K$ and a cyclic vector $\Omega\in K$ such that $K=[\pi(A)\Omega]$ (=closed linear span) and

$$\rho(X)=(\Omega, \pi(X)\Omega)\forall X\in A$$

([8] section 2.3.3). The representation $\pi$ is normal if and only if $\rho$ is a normal state, and then $\pi(A)$ is a W*-algebra as well. This construction is unique up to unitary equivalence. When the normal state $\rho$ has central support $\mathbb{1}$ then the GNS representation is faithful (it is a W*-isomorphism, a relation denoted $\sim$ below). For $A_{\mathscr{S}}=B(H_{\mathscr{S}})$ the normal representations are unitarily equivalent to an amplification ([9] section 2.7)

$$\pi(X)\simeq X\otimes\mathbb{1}.$$

The $A_{\mathscr{S}}$ can be identified with one factor in a tensor product ([9] section 1.22)

$$\begin{aligned} &A_{\mathscr{S}}\sim A_{\mathscr{S}}\otimes\mathbb{1}\subset A\simeq A_{\mathscr{S}}\otimes A_{\mathscr{R}}\\ &A_{\mathscr{R}}\sim\mathbb{1}\otimes A_{\mathscr{R}}\subset A. \end{aligned} \tag{2.1}$$

In this setting the dynamics is given by a $\sigma$-weakly continuous group $T(t)$ (where $T \in \mathbb{R}$ or $\mathbb{Z}$) of W*-automorphisms of $A$. Assume that the state $\rho$ is stationary: $\rho \circ T(t) = \rho$. The stationarity of the state $\rho$ means that in the GNS representation the dynamics is given by a strongly continuous group of unitary operators $W(t)$ in $K$ defined by

$$W(t)\pi(X)\Omega = \pi(T(-t)[X])\Omega \qquad \forall X \in A$$

which leaves $\Omega$ invariant: $W(t)\Omega = \Omega$. The triple $\{\pi(A), W(t), \Omega\}$ is called a *covariant representation* of $\{A, T(t), \rho\}$ ([8] chapter 2.7).

For infinite systems the standard formalism uses a quasi-local C*-algebra, call it $B$, and a strongly continuous representation of $\mathbb{R}$ in the group of C*-automorphisms of $B$. For any invariant state $\rho$ on $B$ there is again a GNS construction where the dynamics is given by a strongly continuous group of unitary operators. The W*-algebra of interest is $A = \pi(B)''$ ($'$ denotes the commutant, $''$ the bicommutant). $A_{\mathscr{S}}$ is now one of the local algebras representing finite parts of the system. If $\rho$ is a locally normal state then $\pi$ restricted to $A_{\mathscr{S}}$ is normal. When $A_{\mathscr{S}} = B(H_{\mathscr{S}})$ then the local normality means that there is again an identification (2.1) for some choice of $A_{\mathscr{R}}$.

In the following we will understand by a *quantum dynamical system* any collection of objects $\{A_{\mathscr{S}}, A, T(\mathbb{R}), \rho\}$ of the structure described above, with $\rho$ stationary. This notation differs from the standard one only by the explicit introduction of $\mathscr{S}$ in order to have an open system formalism. It will be applied to a covariant representation $\{A_{\mathscr{S}}, A, W(\mathbb{R}), \Omega\}$ where $A$ is a W*-algebra acting in $K$ and where we can use the identification (2.1). This is the kind of system which is actually reconstructed in section 6. The prefix W* will be left out almost everywhere.

For a finite system the spectrum of the Hamiltonian is by definition bounded below and has a finite or infinite number of bound states, and (above a certain threshold) a continuous spectrum of unbound states. The Hamiltonian generates a strongly continuous group of unitary operators $U(t)$ in the Hilbert space $H$ of the system and hence defines a group of automorphisms

$$T(t)[X] = U(t)^{+} X U(t) \in A \forall X \in A.$$

The dynamics $W(t)$ in the covariant representation is related to $U(t)$ in the following way: when $U(t) \in A$ we can write

$$W(t) = \pi(U(t))V(t)$$

where the group of unitary operators $V(t) \in \pi(A)'$ is uniquely defined by

$$V(t)\Omega = \pi(U(-t))\Omega.$$

Consider the simplest case $A = B(H)$. There can be a stationary normal state only if the spectrum of $U(t)$ has a discrete part. Then the support of $\rho$ is a projection $P$ belonging to a discrete part of the spectral resolution. Let $k$ index a CON set in $PH$ diagonalizing $\rho$ and $U(t)$ simultaneously

$$\rho = \sum_k p_k |k\rangle\langle k|$$

where $p_k > 0$, $\Sigma_k p_k = 1$, and introduce

$$\Omega = \sum_k \sqrt{p_k} |k\rangle \otimes |k\rangle \in H \otimes PH \equiv K$$

$$\pi(X) = X \otimes P \in B(K) \tag{2.2}$$

$$W(t) = U(t) \otimes U(-t)P \in B(K).$$

Then $\{\pi(A), W(t), \Omega\}$ is a covariant representation with $\Omega$ as a $W(t)$-invariant vector. The spectrum of $W(t)$ differs from that of $U(t)$ only by a discrete component and the continuum part is that of $U(t)$.

We now turn to the introduction of ergodic properties for these dynamical systems. First, there is a condition that the dynamics of $\mathscr{S}+\mathscr{R}$ mixes the system well, allowing us to probe the whole system by interacting with $\mathscr{S}$. It is evident from the definition of the QCK that we cannot hope to get enough information to recover $A$ unless the system is minimal in the sense that

$$\{T(\mathbb{R})[A_{\mathscr{S}}]\}''=A. \tag{2.3}$$

This condition is part of the definition of a generalized K-flow as introduced by Emch [10]. It is clear how (2.3) is to be interpreted in a covariant representation. It turns out that in the construction of section 6 (2.3) is automatically fulfilled by the reconstructed system. Let $B\{A_{\mathscr{S}}, T(\mathbb{R})\}$ be the algebra of polynomials in the operators in the set $T(\mathbb{R})[A_{\mathscr{S}}]$, then the weak closure of $B$ is the LHS of (2.3), i.e. (2.3) is equivalent to

$$A=[B]_w. \tag{2.4}$$

The same relation then holds for any normal representation as

$$\pi(A)=[\pi(B)]_w.$$

If $\pi$ is a normal GNS representation with cyclic vector $\Omega$, then the set $K_0=\pi(B)\Omega$ is dense in the GNS Hilbert space:

$$K=[\pi(A)\Omega]=[\pi(B)\Omega].$$

Consider the decomposition of dynamical systems satisfying (2.3) in a way which commutes with the action of the dynamics and the observations. Introduce the following W*-algebra in the covariant representation

$$M=\{A\cup W(\mathbb{R})\}'=A'\bigcap W(\mathbb{R})'. \tag{2.5}$$

If (2.3) holds we can replace $A$ by $\pi(A_{\mathscr{S}})$. Any projector $Q\in M$ reduces the covariant representation to one acting in the Hilbert space $QK$. The system is indecomposable if and only if

$$M=\mathbb{C}\cdot\mathbb{1} \tag{2.6}$$

holds, which corresponds to the fact that $\rho$ is an extremal invariant (ergodic) state (we say that $Q$ is trivial). There exist several different notions of ergodicity for non-commutative systes. The following definition is consistent with the convex structure of the QCKs introduced below.

*Definition 1.* The dynamical system $\{A_{\mathscr{S}}, A, T(t), \rho\}$ is said to be *ergodic* if both the minimality condition (2.3) and the indecomposablility condition (2.6) hold.

## 3. Quantum correlation kernels

The non-commutative nature of the observations performed on a quantum system is most clearly displayed in the operational approach [11, 12]. By assumption the operations act on $\mathscr{S}$ only. For $A_{\mathscr{S}}=B(H_{\mathscr{S}})$ they are described by normal completely positive (CP) maps. They form a convex cone generated by the maps

$$\{X\mapsto V^{+}XV; X\in A, V\in A_{\mathscr{S}}, \|V\|\leqslant 1\}$$

(which are the extremal rays) by convex combination. Through polarization a set of linear maps is obtained which is linearly generated by elements

$$X \mapsto V^{+} X W$$

where $V, W \in A_{\mathscr{S}}$. In [12] it is outlined how such maps are related to the action of a large class of generalized measurements (called *instruments* there) on the subsystem $\mathscr{S}$ and to the corresponding probabilities. When a sequence of different observations are made on the system at a succession of instants

$$t = (t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n)$$

the operations and the intrinsic dynamics are composed in a time-ordered causal fashion as non-commuting CP maps on the algebra. The relevant probabilities are expressed in terms of a time-ordered quantum correlation kernal (QCK). The following notation is used

$$(X, t) = \{(X_k \in A_{\mathscr{S}}, t_k)\}_1^n. \tag{3.1}$$

The QCK is defined as a sequilinear form on the set of all such time-ordered sequences

$$\begin{aligned} R(X, t | Y, t) &= \rho(V(X, t)^{+} V(Y, t)) \\ V(X, t) &= T(t_n)[X_n] T(t_{n-1})[X_{n-1}] \ldots T(t_1)[X_1] \end{aligned} \tag{3.2}$$

where $\rho$ is the state of the system $\mathscr{S} + \mathscr{R}$ at $t = 0$. It will be assumed stationary in the following and the QCK inherits an obvious stationarity property (5.4). A point of notation: $R$ restricted to $n$-component vectors $(X, t)$ is a QCK $R_n$ of order $n$, and $R$ is then a family $\{R_n; n = 1, 2, \ldots\}$ of compatible kernels (satisfying condition (2) of section 5) for which the notation QCK is used. There is an evident positive definiteness property of the QCK:

$$\sum_{k,l} \lambda_k^* \lambda_l R(X_k, t_k | X_l, t_l) \geqslant 0 \qquad \forall \{(X, t)_k, \lambda_k \in \mathbb{C}\}. \tag{3.3}$$

To this notion of positivity corresponds a natural convex structure and partial order in the set of all stationary QCKs over a given algebra $A_{\mathscr{S}}$. There is an associated notion of an *ergodic* QCK (see (7) of section 5).

By definition the probabilities of any sequence of outcomes of any sequence of observations are given by diagonal elements $R(X, t | X, t)$ of the QCK. Note that the nature of the observations and the space of outcomes at different instants can be quite different and this generality is implicit in the formalism. In fact, the QCK defines the probability distributions for all possible, in general mutually incompatible, sequences of instruments acting on $\mathscr{S}$, thus giving a complete expression for the concept of complementarity. The QCK also describes the interaction of $\mathscr{S}$ with any external system, like a measuring apparatus described by its own quantum dynamics and state, combined with observations performed on the apparatus. Here the dynamics and the initial state of $\mathscr{S} + \mathscr{R}$ is given, everything else arbitrary. This statement follows from the expansion of such an interaction in standard time-dependent perturbation theory. Up to a normalization, any non-negative form (3.3), or any limit of such expressions, can be realized as the probability an outcome of a general measurement of this type. Note that the arguments in the QCK are not necessarily observables but elements of operations on the algebra of observables. For $A_{\mathscr{S}} = B(H_{\mathscr{S}})$ they are in the algebra, though not self-adjoint in general.

On the time ordered sequences (3.1) the following composition operation is introduced. For each pair $(X, t)$, $(Y, u)$ such that $u \geqslant t$ (i.e. $u_k \geqslant t_l \forall k, l$), write

$$(Y, u) * (X, t) = (Y * X, u * t) \tag{3.4}$$

where $Y * X = (X_1, \ldots, X_m, Y_1, \ldots, Y_n)$ etc. Then, from (3.2)

$$V((Y, u) * (X, t)) = V(Y, u) V(X, t). \tag{3.5}$$

## 4. Singular and regular dynamics

An outline of some well known facts is given here [5, 6, 13]. Let $U(t)$ be a strongly continuous group of unitaries in $H$. Let there be a subspace $H_+$ such that

$$\begin{gathered} U(t)H_+ \subseteq H_+ \forall t \in \mathbb{R}_+ \\ \bigcap_{t \in \mathbb{R}_+} U(t)H_+ = \{0\} \end{gathered} \tag{4.1}$$

and introduce the subspace

$$\bigcup_{t \in \mathbb{R}} U(t)H_+ \equiv H_1 \subseteq H.$$

Then, by definition, $\{U(t); t \in \mathbb{R}\}$ is a bilateral shift on $H_1$, while the semigroup $\{U(t)|H_+; t \in \mathbb{R}_+\}$ is a unilateral shift consisting of isometric (non-unitary) transformations. Here the dimensions of $H_+$, $H_1$ and $H_1 \ominus H_+$ are the same, either zero or infinite.

One can always decompose $H$ into $U(t)$-invariant subspaces $H = H_1 \oplus H_2$, such that $U(t)|H_1$ is a shift (the *regular* part) and $U(t)|H_2$ has no shift component (the *singular* part), but this decomposition is not unique. The part $U(t)|H_2$ has the property that for any subspace $H_3 \subseteq H_2$ it holds that

$$\{U(t)H_3 \subseteq H_3 \forall t \in \mathbb{R}_+\} \Rightarrow \{U(t)H_3 \subseteq H_3 \forall t \in \mathbb{R}\}$$

and a continuous group of unitaries with this property can contain no shift. This property is related to the spectrum of $U(t)$ in the folllowing way. Write

$$U(t) = \int_{-\infty}^{\infty} P(\mathrm{d}\omega) \exp(-\mathrm{i}\omega t).$$

For every $\phi \in H$ there is a probability measure on $\mathbb{R}$ given by $\langle \phi | P(\mathrm{d}\omega) | \phi \rangle$. Let $\sigma(\phi, \omega)$ denote the derivative of the absolutely continuous part of the measure. Then the following result holds.

*Lemma 1.* $U(t)$ is a singular if and only if for every $\phi \in H$

$$\int_{-\infty}^{\infty} \mathrm{d}\omega \frac{|\ln \sigma(\phi, \omega)|}{1 + \omega^2} = \infty. \tag{4.2}$$

This is a version of a well known theorem of Kolmogorov and Krein ([5] section III.2, [6] chapter 5.8, [14] Appendix B.12 or [15] section 10.14). In fact, if $U(t)$, $H_+$ satisfy (4.1) then there is a vector $\phi \in H$ such that

$$H_+ \equiv [U(\mathbb{R}_+)\phi] \neq H_1 \equiv [U(\mathbb{R})\phi].$$

But $H_1 \simeq L_2(\mathbb{R}; \sigma(\phi, \cdot))$ so this is equivalent to the statement that the family of functions $\{\exp(i\omega t); t \geqslant 0\}$ is not dense in $L_2(\mathbb{R}; \sigma(\phi, \cdot))$. By the mentioned theorem the family is total precisely when (4.2) holds. It holds for instance if the continuum part of the spectrum does not cover the whole real line. □

The dynamics relevant for the application of these concepts to the reconstruction theorem is that of the covariant representation. It follows from (2.2) and the lines following it and (4.2) that this dynamics will be singular for a finite quantum system. The same conclusion can be drawn for the dynamics of an infinite system when the reference state satisfies the KMS condition for a finite temperature. Let $\alpha_t$ be the dynamical group of automorphisms of the quasilocal algebra $B$. For any two observables $X, Y \in B$ the functions

$$\{f_1(t), f_2(t)\} = \{\rho(\alpha_t(X)Y), \rho(Y\alpha_t(X))\}$$

are the boundary values of a function $f(z)$ holomorphic in the strip $\beta < \operatorname{Im} z < 0$, i.e. it holds that $f_1(t) = f(t), f_2(t) = f(t - i\beta)$ ([8] chapter 5.3). In the covariant representation there is a strongly continuous group $W(t)$ of unitaries such that for every pair $\phi, \psi \in K$,the function $f(t) = \langle\psi| W(t)|\phi\rangle$ can be continued to a function holomorphic in the strip and continuous on the boundary. From the holomorphy and the edge of the wedge theorem ([8] proposition 5.3.6) it follows immediately that if $\phi, \psi \in K$ are such that $f(t) = 0 \forall t > 0$, then $f(z) = 0$ in an open set in the strip and hence it vanishes on the strip and on its boundary, i.e. $f(t) = 0$ for all real $t$. Now let the subspace $K_+$ of $K$ be mapped into itself under positive time translations:

$$W(\mathbb{R}_+)K_+ \subseteq K_+$$

and choose $\phi \in K_+$, $\psi \perp K_+$. Then it is found from the present argument that $K_+$ is invariant also under negative time translations, so that $W(t)$ is singular. It is also possible to use lemma 1 directly to prove this point, compare appendix A.5 of [16]. In this argument it is the holomorphy which is important rather than the KMS boundary condition, hence there is a larger set of states having the desired properties. The set of ground states where the holomorphy extends to a half-plane corresponding to $\beta = \infty$ can be considered as a limiting form of KMS states ([8] chapter 5.3.).

For the uniqueness of the reconstruction in section 6 the following consequence of the singular property of the dynamics is essential. Consider a covariant representation of the type discussed in section 2 and write $T(t)[X] = W(t)^+ X W(t)$. The operator $V(X, t)$ can be defined as in (3.2) if in that formula $X_k \in A_{\mathscr{S}}$ is replaced by the operator $\pi(X_k) \in \pi(A_{\mathscr{S}})$. Then, with the notation $\theta_s t = t - s$

$$T(-s)[V(X, t)] = V(X, \theta_s t).$$

Introduce the following vectors of the GNS Hilbert space $K$

$$\Psi(X, t) = V(X, t)\Omega.$$

From the invariance of $\Omega$ follows that

$$W(s)\Psi(X, t) = \Psi(X, \theta_s t).$$

The set of such vectors with non-positive time parameters span a Hilbert subspace

$$K_- = [\Psi(X, t); X_k \in A_{\mathscr{S}}, t_k \leqslant 0, \forall k] \subseteq K \tag{4.3}$$

which is obviously mapped into itself by $W(t)$ for $t \geqslant 0$, hence by the singular property this holds for all $t$. From the definition it also follows that $\pi(A_{\mathscr{S}})$ maps $K_-$ into itself. In fact, consider elements $(X, t)$ with $t \leqslant 0$. Using (3.4) and (3.5) it is then evident that

$$\pi(Y)\Psi(X, t) = \Psi((Y, 0) * (X, t)) \subseteq K_-$$

and linear combination and closure gives the conclusion. Again let $B$ be the algebra of polynomials in the variables

$$\{\pi(Y, t) = W(t)^+ \pi(Y) W(t);\ t \in \mathbb{R}\}.$$

Then it follows from the inclusion relations above that $B$ maps $K_-$ into itself, and the same holds for the closure (2.4). From this result the following lemma is an immediate consequence.

*Lemma 2.* If $W(t)$ is singular and if $[A\Omega] = K$, then $\Omega$ is cyclic already for the time-ordered products $V(X, t)$ with non-positive time parameters ($t \leqslant 0$), in other words $K_- = K$.

The discussion in this section can be rephrased to hold for a discrete time parameter. In this case there is a spectral resolution in $[0, 2\pi)$ and (4.2) is replaced by

$$\int_0^{2\pi} d\omega\ |\ln \sigma(\phi, \omega)| = \infty.$$

Note that when a continuous time parameter is discretized, the singular property of the dynamics is not preserved in general.

## 5. Properties of the QCK

The time-ordered QCK defined in section 3 satisfies a number of properties which are summarized here. Most of them are immediate consequences of the definitions. The singular property (5.6) is the one which is essential for the sufficiency of the time ordered QCK for a unique reconstruction as described in section 6. This point is where the present scheme differs from those of [4, 7]. First, a convenient equivalence relation is introduced. Using the definition (3.1) we write

$$\{(X, t) \simeq (X', t')\} \Leftrightarrow \{R(X, t | Y, u) = R(X', t' | Y, u) \forall (Y, u)\}.$$

The properties of the QCK are given by (1)–(8) below.

(1) *Positivity.* $R(X, t | Y, u)$ is linear in the $Y_k \in A_{\mathscr{S}}$, conjugate linear in the $X_k \in A_{\mathscr{S}}$ and satisfies the positivity property (3.3). This implies the symmetry $R(X, t | Y, u)^* = R(Y, u | X, t)$. Another straightforward consequence is the following Schwarz type inequality. For any set of time-ordered sequences $\{(X, t)_k, t_k \leqslant 0\}$ and any $Y \in A_{\mathscr{S}}$ it holds that

$$\sum_{k,l} \lambda_k^* \lambda_l R((Y, 0) * (X_k, t_k) | (Y, 0) * (X_l, t_l)) \leqslant \|Y\|^2 \sum_{k,l} \lambda_k^* \lambda_l R(X_k, t_k | X_l, t_l). \quad (5.1)$$

(2) *Compatibility.* From the fact that $T(t)[\mathbb{1}] = \mathbb{1}$ follows: If $(X, t)$ can be transformed into $(X', t')$ through the addition and deletion of dummy arguments of the form $(\mathbb{1}, t_k)$ then $(X, t) \simeq (X', t')$.

(3) *Normalization.* From the normalization of the reference state follows that, with the notation $(1)=(\mathbb{1},\mathbb{1},\ldots,\mathbb{1})$

$$R(1,t|1,u)=1. \tag{5.2}$$

(4) *Multiplicativity.* From the fact that $T(0)$ is the identity map follows:

(a) if $(X,t)$ is such that $t_m=t_{m+1}$ for some $m$, and if $X'$ is obtained from $X$ by the replacement

$$(X_m,X_{m+1})\mapsto(X'_m,X'_{m+1})=(X_{m+1}X_m,\mathbb{1})$$

then $(X,t)\simeq(X',t)$.

(b) For all $(X,t)$, $(Y,u)$ *with* $t,u\leqslant 0$, *and all* $Z\in A_{\mathscr{S}}$ it holds that

$$R((Z,0)*(X,t)|Y,u)=R(X,t|(Z^{+},0)*(Y,u)). \tag{5.3}$$

(5) *Stationarity.* From the stationarity of the reference state follows: for all $(X,t)$, $(Y,u)$ and all real $s$

$$R(X,\theta_s t|Y,\theta_s u)=R(X,t|Y,u). \tag{5.4}$$

(6) *Continuity.* From the (local) normality of the reference state follows that each of the maps $Y_k\mapsto R(X,t;Y,u)$ is $\sigma$-strongly continuous. From the continuity of the dynamics follows that for all $(X,t)$, $(Y,u)$

$$\lim_{s\to 0}R(X,t|Y,\theta_s u)=R(X,t|Y,u). \tag{5.5}$$

(7) *Ergodicity.* We call $R$ *ergodic* if it is an extremal element in the convex set defined by (1)–(6) for a given algebra $A_{\mathscr{S}}$. The QCK of an ergodic dynamical system is ergodic in this sense. This is proved under (e) of section 6.

(8) *Singularity and regularity.* If the dynamics of the covariant representation associated with $\rho$ is singular, then by lemma 2 it holds that for every $(Y,u)$ the infimum over the finite sets $\{(X,t)_k;t_k,\leqslant 0\}$:

$$\inf\{R(Y,u|Y,u)+\sum_{k,l}R(X_k,t_k|X_l,t_l)-\sum_k R(X_k,t_k|Y,u)-\sum_l R(Y,u|X_l,t_l)\}=0. \tag{5.6}$$

We call $R$ *singular* if (5.6) holds. On the other hand we call $R$ *regular* if for every singular QCK $R_1$ the inequality $pR_1\leqslant R$, $(p\geqslant 0)$, implies $p=0$. Instead of singular and regular we could use the words *deterministic* and *non-deterministic* as will be clear from the developments in section 6.

The properties given above are not independent or minimal. From (5.1) follows: for all $Y\in A_{\mathscr{S}}$ it holds that

$$\{(X,t)\simeq(X',t');tt'\leqslant 0\}\Rightarrow\{(Y,0)*(X,t)\simeq(Y,0)*(X',t')\}. \tag{5.7}$$

If it is assumed that for all $(X,t)$ with $t\leqslant 0$

$$(X,t)\cong(\mathbb{1},0)*(X,t) \tag{5.8}$$

then the compatibility conditions follow from this and (5.7). In the same way, from the following relation, for all $Y_1$, $Y_2\in A_{\mathscr{S}}$, all $(X,t)$ with $t\leqslant 0$:

$$(Y_1,0)*((Y_2,0)*(X,t))\simeq(Y_1Y_2,0)*(X,t) \tag{5.9}$$

and (5.8) the multiplicativity property (4a) follows.

## 6. Reconstruction and determinism

In this section we consider the convex set of time-ordered QCKs over a fixed algebra $A_{\mathscr{S}}$, each one having the properties specified in section 5.

*Theorem 1.* The QCK $R$ can be decomposed in a unique way into a convex combination of a singular and a regular QCK. From the singular part we can reconstruct a dynamical system which is unique up to unitary equivalence. If a singular QCK derives from a covariant representation which is minimal in the sense of (2.3), then the reconstructed system is equivalent to this representation. The reconstruction starting from a regular QCK always involves an arbitrariness which implies that the full set of correlations of arbitrary time order is not uniquely determined.

The proof involves some steps which are standard (compare theorem 3.7 of [3], chapter 1 of [17]), but they are sketched here for completeness. Steps (c) and (f) are the important points which allow us to go further than [4,7].

(a) *Hilbert space.* On the linear span of formal elements $K_0=\{\mathbf{\Phi}(X,t)\}$ a pre-Hilbert structure is introduced through

$$\langle\mathbf{\Phi}(X,t)|\mathbf{\Phi}(Y,u)\rangle=R(X,t|Y,u).$$

The null space is

$$N=\{\phi\in K_0;\langle\phi|\phi\rangle=0\}$$

and the Hilbert space $K$ is the completion of the set of equivalence classes of $K_0$ modulo $N$: $K=[K_0/N]$. We use the following notation for the elements of $K_0/N$

$$\mathbf{\Psi}(X,t)=\mathbf{\Phi}(X,t)\bmod N.$$

(b) *Dynamics.* A map $W_0(s)$ of $K_0$ into itself is defined by

$$W_0(s)\mathbf{\Phi}(X,t)=\mathbf{\Phi}(X,\theta_s t).$$

From (5.4) follows that it maps the null space $N$ into itself, hence the following definition of the dynamics makes sense on $K_0/N$:

$$W(s)\mathbf{\Psi}(X,t)=\mathbf{\Psi}(X,\theta_s t).$$

Furthermore, $W$ is isometric on $K_0/N$ and can be continued to an isometry on $K$. It is evident that it forms a group of transformations, hence it is a group of unitary operators in $K$. From (5.5) follows that

$$\lim_{s\to 0}\|W(s)\psi-\psi\|=0\;\forall\psi\in K_0/N.$$

From this and the isometric property it then follows that the same holds for all $\psi\in K$. The subspace $K_-$ is defined by (4.3) and $K_{-\infty}$ by

$$K_{-\infty}=\bigcap_{s\in\mathbb{R}}W(s)K_-\subseteq K_-.$$

From (a) follows that

$$K=\bigcup_{s\in\mathbb{R}}W(s)K_-.$$

The subspace $K_{-\infty}$ is invariant under $W(\mathbb{R})$, and in $K\ominus K_{-\infty}$ the dynamics acts as a bilateral shift.

(c) *Representation of* $A_{\mathscr{S}}$. First consider the singular case. Then (5.6) implies that $K=K_-$. For each $(X, t)$, $t\leqslant 0$, $Y\in A_{\mathscr{S}}$, define $\pi_0(Y)$ through

$$\pi_0(Y)\Phi(X, t)=\Phi(Y*X, 0*t).$$

From (5.1) follows that $\pi_0(Y)$ maps $N$ into itself and that

$$\|\pi_0(Y)\Phi(X, t)+N\|\leqslant\|Y\|\cdot\|\Phi(X, t)\|.$$

Hence $\pi_0(Y)$ extends by continuity to an operator $\pi(Y)$ in $K$ with norm $\|\pi(Y)\|\leqslant\|Y\|$. From (5.9) follows that $\pi(X)\pi(Y)=\pi(XY)$, from (5.3) that $\pi(Y)^+=\pi(Y^+)$, and from (5.8) that $\pi(\mathbb{1})=\mathbb{1}$. From the $\sigma$-strong continuity of the QCK in the operator arguments follows that $\pi$ is a normal representation. With $A_{\mathscr{S}}=B(H_{\mathscr{S}})$ it is unitarily equivalent to one of the form $\pi(X)=X\otimes\mathbb{1}$. Now let there be a regular part in the QCK, i.e. the infimum in (5.6) is not identically zero. Together with the stationarity this means that $K\neq K_-$, but there is still a representation $\pi_-(A_{\mathscr{S}})$ in $K_-$ as described above. In general $\pi_-(A_{\mathscr{S}})$ does not leave $K_{-\infty}$ invariant. As $K_+=K\ominus K_-$ is always of infinite dimension (if non-zero), we can introduce a normal non-degenerate represenation $\pi_+(A_{\mathscr{S}})$ in this subspace which is again of the simple form (2.1). Thus there is a representation $\pi(A_{\mathscr{S}})$ on $K$ which is the direct sum of $\pi_-$ and $\pi_+$.

(d) $W^*$*-algebra.* Using $W(t)$, $\pi(A_{\mathscr{S}})$ in (c), define

$$\Omega=\Psi(1, t)$$

$$\pi(X, t)=W(t)^+\pi(X)W(t)$$

$$\pi(\mathbf{X}, \mathbf{t})=\pi(X_n, t_n)\ldots\pi(X_2, t_2)\pi(X_1, t_1).$$

It follows from property (2) in section 5 that $\Omega$ is a well-defined element of $K$, and from (5.2) that it has unit norm. From the definition it also follows that $\Omega$ is invariant under $W(t)$, hence $\Omega\in K_{-\infty}$, and that

$$\Psi(\mathbf{X}, \mathbf{t})=\pi(\mathbf{X}, \mathbf{t})\Omega.$$

Furthermore, for all sets $\{(\mathbf{X}, \mathbf{t}), n, t_n\leqslant t\}$ it holds that

$$W(t)\Psi(\mathbf{X}, \mathbf{t})\in K_-$$

and from this follows by recursion that the time ordered products $\Psi(\mathbf{X}, \mathbf{t})$ are determined by $W(\mathbb{R})$ and $\pi_-(A_{\mathscr{S}})$ and are independent of the choice of $\pi_+(A_{\mathscr{S}})$. The same then holds for the time-ordered QCK. However, the choice of $\pi_+(A_{\mathscr{S}})$ will enter into the W*-algebra of the system as defined in the fashion of (2.3):

$$A=\{\pi(A_{\mathscr{S}}, t); t\in\mathbb{R}\}''.$$

Here $\pi(A_{\mathscr{S}})$ is a subalgebra of the form described above and there is a unitary equivalence $A\simeq A_{\mathscr{S}}\otimes A_{\mathscr{R}}$ for some choice of $A_{\mathscr{R}}$.

(e) *Ergodicity.* First consider the case where $\boldsymbol{R}$ is singular and let there be a QCK $\boldsymbol{R}_1$ with $p\boldsymbol{R}_1\leqslant\boldsymbol{R}$ for some $0<p\leqslant 1$. By a standard argument (theorem 1.12 of [17]) there is a bounded map betwen the Hilbert spaces $V: K\to K_1$ satisfying $V^+V\leqslant p^{-1}\mathbb{1}$ and

$$\Psi_1(\mathbf{X}, \mathbf{t})=V\Psi(\mathbf{X}, \mathbf{t})$$

$$\boldsymbol{R}_1(\mathbf{X}, \mathbf{t}\,|\,\mathbf{Y}, \mathbf{u})=\langle\Psi(\mathbf{X}, \mathbf{t})|\,V^+V\Psi(\mathbf{Y}, \mathbf{u})\rangle.$$

From (5.3) and (5.4) follows that $Q=V^+V$ satisfies, for all $\{X\in A_{\mathscr{S}}, t\in\mathbb{R}\}$

$$[Q, \pi(X, t)]=[Q, W(t)]=0$$

i.e. $Q$ belongs to the algebra (2.5) and the condition (2.6) is broken if $Q$ is non-trivial. Conversely, if the QCK $\boldsymbol{R}$ is ergodic, then $Q$ is trivial and (2.6) holds. Now consider the case where $\boldsymbol{R}$ is not singular. There is then a non-negative operator $Q$ in $K$ which satisfies $[Q, W(t)]=0$, $QK_- \subseteq K_-$, and

$$[Q, \pi_-(A_{\mathscr{S}})]|K_- = 0.$$

By Hilbert space duality $QK_+ \subseteq K_+$. It is actually enough to consider the case where $Q$ is a projection. On the spaces $Q(K \ominus K_\infty)$ and $(1-Q)(K \ominus K_\infty)$ there are defined bilateral shifts, hence it follows that the subspaces $QK_+$ and $(1-Q)K_+$ are either null or of infinite dimension. It is then always possible to define $\pi_+(A_{\mathscr{S}})$ such that

$$[Q, \pi_+(A_{\mathscr{S}})]|K_+ = 0$$

and consequently $Q$ is in (2.5). Thus the decomposition has been extended from the causal QCK to a subset of the arbitrary choices of the reconstructed system to recover the situation obtained in the singular case. Note that if $Q|K_-$ is trivial then so is $Q|K_+$. Hence, if the causal QCK is indecomposable then so is the whole dynamical system, and this proves (7) of section 5.

(f) *Regular and singular parts.* Define a Hilbert subspace and the corresponding projector

$$K_r = P_r K = [A(K \ominus K_{-\infty})]$$

and the complementary subspace $K_s = K \ominus K_r = P_s K$. Due to the fact that $K \ominus K_{-\infty}$ is invariant under $W(\mathbb{R})$ it holds that $K_r$ is the smallest subspace of $K$ containing $K \ominus K_{-\infty}$ which is left invariant by both $\pi(A_{\mathscr{S}})$ and $W(\mathbb{R})$, and hence by the algebra generated by these two sets (this is the commutant of (2.5)). By duality in Hilbert space, $K_s$ is the largest subspace of $K_{-\infty}$ with this invariance. But

$$\pi(A_{\mathscr{S}})|K_{-\infty} = \pi_-(A_{\mathscr{S}})|K_{-\infty}$$

hence $K_s$ (and $K_r$) is uniquely defined by $W(\mathbb{R})$ and $\pi_-(A_{\mathscr{S}})$ and independent of the choice of $\pi_+(A_{\mathscr{S}})$. The same holds for the projectors $P_s$, $P_r$. Furthermore, they are in the set (2.5) and define a unique decomposition of the QCK into a convex combination

$$\boldsymbol{R} = \langle \Omega | P_s \Omega \rangle \boldsymbol{R}_s + \langle \Omega | P_r \Omega \rangle \boldsymbol{R}_r$$

where, for instance.

$$\langle \Omega | P_s \Omega \rangle \boldsymbol{R}_s(X, t | Y, u) = \langle \Psi(X, t) | P_s \Psi(Y, u) \rangle.$$

(g) *Unicity.* In the singular case ($P_s = \mathbb{1}$) any two constructions (indexed 1,2) based on the given QCK are unitarily equivalent. In fact, the relation

$$V\Psi_1(X, t) = \Psi_2(X, t)$$

defines an isometric bijection $V: K_1 \to K_2$ between the two Hilbert spaces as is evident from the identity of the QCKs and the cyclic property of the construction. The same conclusion holds for any covariant representation satisfying (2.3) and having the given singular QCK. When there is non-trivial regular part, then the arbitrariness in defining the representation $\pi_+(A_{\mathscr{S}})$ means that there is for every unitary $V$ in $K_+$ a transformation

$$\{\pi_-(X), \pi_+(X), W(t)\} \mapsto \{\pi_-(X), V^+ \pi_+(X) V, W(t)\}$$

which leaves invariant the time-ordered QCK but it is not a unitary equivalence for the dynamical system unless $V$ and $W(t)$ commute. On the other hand, the reconstruction theorem of [4] shows that the QCK of arbitrary time order gives a reconstruction which is unique up to unitary equivalence. Thus, when there is a regular part we can find many continuations from the QCK to the full set of correlations by choosing $V$ arbitrarily. □

From the proof of the point (c) above it follows that singular systems have a deterministic property (predictability) which is analogous to that of deterministic stationary stochastic processes, with due consideration of the non-determinism which is inherent in quantum theory.

*Theorem* 2. Let there be given a dynamical system with a singular QCK. For any $t>0$ and any $\phi \in H_{\mathscr{S}}$ there is then a limit of sequences of linear combinations of elements $\{(X, t)_k; t_k \leqslant 0\}$ such that the limit represents the pure state $\phi$ on the observables of $\mathscr{S}$ at time $t$

$$\lim \sum_{k,l} \lambda_k^* \lambda_l R((Y, t) * (X, t)_k | (Y, t) * (X, t)_l) = \langle \phi | Y^+ Y \phi \rangle.$$

In other words, the preparation of a suitable state of the system at $t=0$ by operations acting on $\mathscr{S}$ during $(-\infty, 0]$ allows us to make an as well determined prediction of the outcome of any given observation of $\mathscr{S}$ at a given time $t>0$ as the quantum theory of closed systems allows. Conversely, if this predictability holds for any choice of $\phi \in H_{\mathscr{S}}$ and for all $t \in [0, r]$, some $\tau > 0$, then the QCK must be singular.

The proof of the first part follows directly if we observe that there is a non-trivial subspace

$$\pi(|\phi\rangle\langle\phi|)K \subset K$$

and that any normalized vector $\Psi$ in it will give the expectation $\langle \phi | X | \phi \rangle$ for $\pi(X), X \in A_{\mathscr{S}}$. Now $W(-t)\Psi \in K$ will give this expectation for $\pi(A_{\mathscr{S}}, t)$ and as $K=K_-$ the first statement follows. For the converse statement we note that the predictability holds for time $t$ if and only if

$$\pi(|\phi\rangle\langle\phi|) W(t) K_- \subseteq W(t) K_- .$$

This holds for all $\phi \in H_{\mathscr{S}}$ if and only if

$$\pi(A_{\mathscr{S}}) W(t) K_- = W(t) K_-$$

which is equivalent to

$$\pi(A_{\mathscr{S}}, t) K_- = K_- .$$

If this holds for all $t \in [0, \tau]$, then we find from (a) and (b) that $W(\tau)^+ K_- = K_-$, and consequently that

$$K = \lim_{t \to \infty} W(t)^+ K_- = K_-$$

which means that the QCK is singular. □

## 7. Markov processes

It will now be shown that a Markov quantum stochastic process (QSP), as defined in a standard way, necessarily has a shift in the dynamics of the reconstructed system unless the dynamics on $A_{\mathscr{S}}$ is an automorphism group, which corresponds to the dynamics of a closed finite system. Consequently, for a Markov process describing a genuine open system dynamics, the time-ordered QCK does not suffice for a unique reconstruction in the sense used here. There is a considerable number of papers on the reconstruction (dilation) problem in the Markov case [17–23]. There the reader can find many aspects on the dilation problem which have been left out here.

The generally accepted definition of a Markov process in the non-commutative case starts from a semigroup $\{T_{\mathscr{S}}(t); t \in \mathbb{R}_+\}$ of normal unital CP maps on $A_{\mathscr{S}}$ [24]

$$T_{\mathscr{S}}(t+u) = T_{\mathscr{S}}(t) \cdot T_{\mathscr{S}}(u)$$

$$T_{\mathscr{S}}(t)[\mathbb{1}] = \mathbb{1}$$

$$T_{\mathscr{S}}(0)[X] = X \forall X \in A_{\mathscr{S}}.$$

It is assumed to be $\sigma$-weakly continuous in $t$ and to have a normal stationary state

$$\rho_{\mathscr{S}} \cdot T_{\mathscr{S}}(\mathbb{R}_+) = \rho_{\mathscr{S}}.$$

In order to specify the Markov property one must prescribe how the QCK is generated from the semigroup. For $n$-vectors of the form (3.1), the $n$th order kernel is defined recursively by the formula, sometimes called the *quantum regression theorem* [12],

$$\begin{aligned} &R_n(\boldsymbol{X}, \boldsymbol{t} \mid \boldsymbol{X}, \boldsymbol{t}) = \rho_{\mathscr{S}}(\hat{R}_n(\boldsymbol{X}, \boldsymbol{t})_1^n) \\ &\hat{R}_m(\boldsymbol{X}, \boldsymbol{t})_1^m = X_1^+ T_{\mathscr{S}}(t_2 - t_1)[\hat{R}_{m-1}(\boldsymbol{X}, \boldsymbol{t})_2^m] X_1 \qquad (7.1) \\ &\hat{R}_1(\boldsymbol{X}, \boldsymbol{t}) = X^+ X. \end{aligned}$$

The general element is obtained by polarization. The kernels of all orders define a QCK satisfying the properties (1)–(6) of section 5. Let there be a representation of this process of the form given in section 6, eventually obtained using also QCK elements of unphysical time order. Thus, with the notation of section 6 there is an equality

$$R(\boldsymbol{X}, \boldsymbol{t} \mid \boldsymbol{Y}, \boldsymbol{u}) = \langle \Psi(\boldsymbol{X}, \boldsymbol{t}) | \Psi(\boldsymbol{Y}, \boldsymbol{u}) \rangle$$

for all time-ordered arguments, where the kernel has the form (7.1). We will see that there is a strict inclusion in (4.3) in this case. Assume the contrary, namely that for all $\{\psi \in K_-, X \in A_{\mathscr{S}}, t \geqslant 0\}$ it holds that

$$\phi = \pi(X, t)\psi \in K_- \qquad (7.2)$$

and prove a contradiction. The representation of the algebra $A_{\mathscr{S}}$ can again be taken to be of the form

$$\pi(X)=X\otimes\mathbb{1}. \tag{7.3}$$

The relation (7.2) implies that

$$|\langle\phi|\pi(X,t)\psi\rangle|^2=\|\phi\|^2\langle\psi|\pi(X^+X,t)\psi\rangle$$

but (7.1) and (7.3) mean that this identity can be written in the form

$$|\langle\phi|(T_{\mathscr{S}}(t)[X]\otimes\mathbb{1})\psi\rangle|^2=||\phi||^2\langle\psi|(T_{\mathscr{S}}(t)[X^+X]\otimes\mathbb{1})\psi\rangle. \tag{7.4}$$

Now the Schwarz inequality says that in (7.4)

$$\text{LHS}\leqslant||\phi||^2\langle\psi|(T_{\mathscr{S}}(t)[X]^+T_{\mathscr{S}}(t)[X]\otimes\mathbb{1})\psi\rangle.$$

But the Schwarz inequality for CP maps with $T[\mathbb{1}]=\mathbb{1}$ reads [24]

$$T[X]^+T[X]\leqslant T[X^+X]$$

with equality if and only if $T$ is an automorphism. If equality holds in (7.4) for all $\psi\in K_-$, i.e. for all vectors in $K$ by our assumption, then it must hold for all $\{X\in A_{\mathscr{S}},\, t\geqslant 0\}$ that

$$T_{\mathscr{S}}(t)[X]^+T_{\mathscr{S}}(t)[X]=T_{\mathscr{S}}(t)[X^+X]. \tag{7.5}$$

This relation means that $X\mapsto T_{\mathscr{S}}(t)[X]$ is a normal represenation of $A_{\mathscr{S}}$, and from the continuity in $t$ follows that it must be simply a unitary equivalence $X\mapsto U(t)^+XU(t)$. We conclude that with the exception of this trivial case a contradiction has been obtained which shows that $K_-\neq K$. One can sharpen this result as follows.

*Theorem 3.* A Markovian system defines a QCK which is either singular, in which case the unitarity (7.5) holds, or regular. Furthermore, the QCK is ergodic if and only if $\rho_{\mathscr{S}}$ is an extremal invariant state for the semigroup.

For the proof of the first part we note that if the projector $P_s$ in (f) of section 6 is non-zero, it holds that

$$\pi(T_{\mathscr{S}}(t)[A])|K_s=W(t)^+\pi(A)W(t)|K_s.$$

Again one finds that (7.5) must hold, consequently that $P_s=\mathbb{1}$. For a regular system the simplest case is that where the dynamics consists of a two-sided shift plus the invariant vector but has no singular part apart from the stationary state. This is equivalent to $K_{-\infty}=\{\Omega\}$, which happens precisely when there is convergence to the stationary state: for all $X$

$$\lim_{t\to\infty}T_{\mathscr{S}}(t)[X]=\rho_{\mathscr{S}}(X)\mathbb{1}.$$

In all other cases the dynamics of the reconstructed system has a singular part in $K_{-\infty}$, even though the QCK has no singular part. This singular part of the dynamics can be completely trivial, however, if there is just a multiplicity of stationary states. For the proof of the second statement it is evident that if $\rho_{\mathscr{S}}$ is decomposed in a non-trivial way into stationary states, then there is a corresponding decomposition of the QCK. Conversely, if $\boldsymbol{R}$ is decomposable then there is a non-trivial projector $P\in A'\cap W(\mathbb{R})'$ such that there is a strict containment $PK\subset K$. There is then a stationary state $\mu_s$ defined by

$$\mu_{\mathscr{S}}(X)=||P\Omega||^{-2}\langle\Omega|\pi(X)P\Omega\rangle.$$

Let $K_1 = [\pi(A_{\mathscr{S}})\Omega]$. Now $\mu_{\mathscr{S}} = \rho_{\mathscr{S}}$ implies that

$$PK_1 = K_1.$$

But if this equality holds we can commute through the operators to find that

$$K = [AK_1] = P[AK_1].$$

This is a contradiction which hows that if the QCK is decomposable, then so is $\rho_{\mathscr{S}}$. The conditions for extremality of $\rho_{\mathscr{S}}$ have been explored by Frigerio [25]. □

It is clear from the results of section 4 and this section that the occurrence of a shift in the dynamics is inconsistent with having a reservoir in thermal equilibrium with a finite temperature. Consequently such heat baths give rise to non-Markovian evolutions and non-exponential decay [26, 27]. This fact can be expressed as a lack of a quantum white noise at a finite temperature. A reservoir will give a quantum dynamical semigroup without approximation only if it is of a very particular type which corresponds to an infinite temperature [28, 29].

## 8. Complements

It was noted already in the introduction that there is a similarity between the theory presented here and that of stationary stochastic processes [5, 6]. However, there are characteristic differences between the two cases. In the commutative case the decomposition into singular and regular components of the process corresponds exactly to the properties of the unitary operators representing the time translation. In the quantum case this is not so in general. The regular component can have a singular part in the dynamics, although there is no singular subprocess. This comes from the fact that the operations defining the QCK allows the outside observer to change the state of the system, he is not restricted to using only the intrinsic dynamics. This fact also means that the decomposition depends on the choice of $A_{\mathscr{S}}$, not just on the dynamics. If we start from a given canonical representation $\{A, W(\mathbb{R}), \Omega\}$ then $K_-$ and $K_{-\infty}$ increase with $A_{\mathscr{S}}$, but it is not clear if it is possible to tell anything in general about $K_s$.

In the commutative case the full statistics of the sample paths give a unique reconstruction, for singular and regular processes. However, this fact does not exclude that there may be minimal non-commutative reconstructions based on the time-ordered correlation kernel of a commutative regular process, and this possibility makes the commutative and non-commutative results consistent. Of course, in the commutative case not only is $A_{\mathscr{S}}$ commutative, but there is an additional property of the kernel saying that summation over the outcomes at any instant gives the same result as not making any observation on the system. In the commutative case a reconstruction of an ergodic stationary process from the complete past of a single sample path is possible for almost all sample paths, both for deterministic and non-deterministic processes [30]. In the quantum case there can be no real counterpart of this result. The observations of the system perturb the reference state, so the stationarity of the state is a weaker concept. Generally we need an infinite ensemble of sample paths which correspond to different choices of instruments in each instant.

In the quantum case the deterministic property involves the preparation of a well-defined state through operations on $\mathscr{S}$ acting in the past. This preparation procedure depends on the quantum state for $\mathscr{S}$ we want to achieve. In the commutative case there is an optimal linear predictor acting on the full past of the process which does not have this dependence. This is just another aspect of the difference between the order structures of quantum and classical measurements. A similar reformulation is necessary also in the case of a closed system $\mathscr{S}$.

An interesting unsolved problem concerns the role the time order may have in defining entropy measures of randomness. The generalization of the Kolmogorov–Sinai entropy introduced by Connes, *et al* [31, 32] does not refer to the causal time order of real observations on the system. Hudetz [33] attempts to create an alternative formalism which takes the time order into account while an earlier approach [34] which defines a dynamical entropy directly from the time-ordered QCK has not been developed to a complete theory. Recently it has been proved that for non-commutative systems with a discrete time parameter a singular spectrum imples zero entropy [35]. Note that a *singular spectrum implies that the dynamics is singular, but it is a much more restrictive* condition. Of course, this result is consistent with the classical case. Moreover, as we have seen above, in this case the time order is unimportant. By analogy with the classical case one expects that the entropy will be zero unless there is a shift of infinite multiplicity in the dynamics [36].

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